Dynamics of Bass-Reflex Loudspeaker Systems (3)

Deriving Equations of Motion of Bass-Reflex Speaker Systems

Shigeru Suzuki

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3. Deriving Equations of Motion

We have seen classification of bass-reflex loudspeaker systems in the previous paper. We will review how to derive those equations of motion in this chapter.

3.1 Deriving Characteristic Frequency of Helmholtz's Cavity Resonator

Bass-reflex loudspeaker cabinet is an application of Helmholtz's cavity resonator. It has characteristic frequencies. Refer to MCAP001E for more details.

State equation of ideal gas under adiabatic condition is expressed in equation (1).

$$PV^{\gamma} = constant \qquad (1)$$

where,

P : Pressure of air in chamber $[N]$

V : Volume of chamber $[m^3]$

γ : Ratio of specific heats $\frac{C_p}{C_v}$.

Arranging equation (1), we get

$$d(PV^{\gamma}) = V^{\gamma} dP + P\gamma V^{\gamma-1} dV = 0$$

Multiplying both members of above equation by $V^{-\gamma+1}$,we get

$$VdP + \gamma PdV = 0 \quad .$$

Therefore,

$$dP = -\frac{\gamma PdV}{V} \qquad (2).$$

Hooke's low is expressed in equation (3).

$$dF = -kdx \qquad (3)$$

where,

k : spring constant of air in chamber $[N/m]$

F : restoring force by air spring $[N]$

There is the following relationship expressed in equation (4).

$$\begin{cases} dF = adP \\ dV = adx \end{cases} \tag{4}$$

where,

a : cross-sectional area of duct $[m^2]$

Substituting equation (4) to equation (3), we get

$$dF = -\frac{\gamma aP}{V} dV = -\frac{\gamma a^2 P}{V} dx \quad .$$

Hence spring constant of air in the chamber under adiabatic condition is expressed as follows:

$$k = \frac{\gamma a^2 P}{V} \tag{5}$$

If we assume isothermal condition[1], we have equation (5)'.

$$k = \frac{a^2 P}{V} \tag{5)'$$

As a result, we can calculate characteristic frequency of Helmholtz's cavity resonator as following equations:

Adiabatic condition:

$$f_D = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{\gamma aP}{\rho lV}} \quad [Hz] \tag{6}$$

Isothermal condition:

$$f_D = \frac{1}{2\pi}\sqrt{\frac{aP}{\rho lV}} \quad [Hz] \tag{6)'$$

where,

$m = \rho al$

m : Mass of air involved in duct $[g]$

ρ : Density of air at room temperature $[kg/m^3]$

l : Length of duct $[m]$

1 Assuming adiabatic condition is theoretically correct; however, assuming isothermal condition resulted in more reasonable result, based on author's experience. This discrepancy may be due to other reasons.

3.2 Deriving Equations of Motion of MCAS-CR (Multiple-Chamber Aligned in Series Cavity Resonator)

Equation (5) shows that spring constant if a function of cross-sectional area. Therefore, we use spring constant for reference cross-sectional area. Definitions of symbols are given in Table 3 of MCAP004E.

Table 3 of MCAP004E Definitions of Symbols

Symbol	**Definition**	**Note**
a_0	Reference area $[m^2]$	Effective area of vibrating membrane
a_j	Cross-sectional area of each duct $[m^2]$	
r_j	Ratio of areas a_j/a_0	
m_j	Mass of bulk air involved in each duct $[g]$	
ρ	Density of air at room temperature $[kg/m^3]$	$\rho=1.2[kg/m^3]$
l_j	Effective length of each duct $[m]$	
P	Atmospheric pressure at room temperature $[Pa]$	$P=101{,}300[Pa]$
k_j	Spring constant of chamber for reference area $[N/m]$	$k_j=\frac{a_0^2 P}{V_j}[N/m]$
V_j	Volume of each chamber $[m^3]$	

Fig.3 in MCAP004E shows schematics of MCAS-CR with n chambers. Applied force to masses m_2 through n_{n-1} is function of volumes of chambers next to the mass and displacement of the mass and next masses. Fig.6 focuses on m_2 through n_{n-1} .

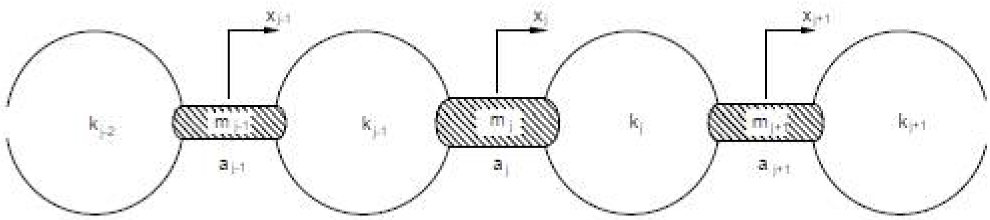

Fig.6

Coss-sectional area of m_j is a_j . Forces from k_{j-1} side and from k_j side, while m_j moves by x_j , are respectively F_{j-1} and F_j in the following equations:

$$F_{j-1} = -k_{j-1}\frac{a_j^2}{a_0^2}x_j = -k_{j-1}r_j^2x_j$$

$$F_j = -k_j\frac{a_j^2}{a_0^2}x_j = -k_jr_j^2x_j$$

where we suppose other masses do not move. Effects of other masses will be considered later superposing effects.

Next let displacement of m_j is zero ($x_j=0$) and displacements of m_{j-1} and m_{j+1} are, respectively, x_{j-1} and x_{j+1} . Let F'_{j-1} and F'_j be forces from side k_{j-1} and k_{j+1} , then

$$F'_{j-1} = k_{j-1}\frac{a_j^2}{a_0^2}\frac{a_{j-1}}{a_j}x_{j-1} = k_{j-1}\frac{a_{j-1}}{a_0}\frac{a_j}{a_0} = k_{j-1}r_{j-1}r_jx_{j-1}$$

$$F'_j = k_j\frac{a_j^2}{a_0^2}\frac{a_{j+1}}{a_j}x_{j+1} = k_j\frac{a_j}{a_0}\frac{a_{j+1}}{a_0} = k_jr_jr_{j+1}x_{j+1}$$

Superposing above equations since these are linear differential equations, we get equation (7) as follows:

$$m_j\frac{d^2x_j}{dt^2} + k_{j-1}r_j^2x_j + k_jr_j^2x_j - k_{j-1}r_{j-1}r_jx_{j-1} - k_jr_jr_{j+1}x_{j+1} = 0$$

videlicet,

$$m_j\frac{d^2x_j}{dt^2} - k_{j-1}r_{j-1}r_jx_{j-1} + (k_{j-1}+k_j)r_j^2x_j - k_jr_jr_{j+1}x_{j+1} = 0 \qquad (7)$$

We have got equation of motion for m_j . Then we will derive equations of motion of masses of both ends.

We focus on m_0 and m_1 in Fig.3 of MCAP004E then draw Fig. 7.

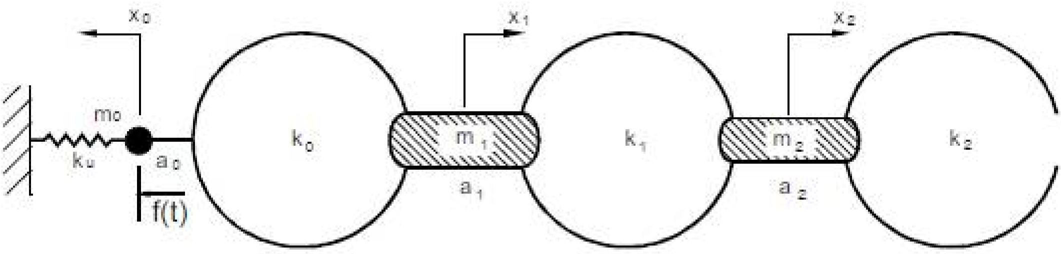

Fig.7

Mass m_0 is equivalent mass of membrane. This mass is forced to vibrate by power amplifier. We note that direction of m_0 is different from other masses' then we derive equation (8) as in the same manner above.

$$m_0 \frac{d^2 x_0}{dt^2} + k_u x_0 + k_0 x_0 + k_0 \frac{a_1}{a_0} x_1 = f(t)$$

videlicet,

$$m_0 \frac{d^2 x_0}{dt^2} + (k_u + k_0) x_0 + k_0 r_1 x_1 = f(t) \tag{8}$$

In the same manner, we get equation of mass m_1 as below:

$$m_1 \frac{d^2 x_1}{dt^2} + k_0 r_1 x_0 + (k_0 + k_1) r_1^2 x_1 - k_1 r_1 r_2 x_2 = 0 \tag{9}$$

At last, we derive equations of motion of mass m_n . Let us note that m_{n+1} does not exist, then equation of motion becomes as equation (10). Also we note that volume of room is much greater than the chamber: $k_n = 0$.

$$m_n \frac{d^2 x_n}{dt^2} - k_{n-1} r_{n-1} r_n x_{n-1} + (k_{n-1} + k_n) r_n^2 x_n = 0 \tag{10}$$

Equation of motion of all masses in the MCAS-CR system is expressed in matrix form as follows:

$$\boldsymbol{M} \frac{d^2 \boldsymbol{X}}{dt^2} + \boldsymbol{Kx} = \boldsymbol{f} \tag{11}$$

where,

$$\boldsymbol{M} = \begin{bmatrix} m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_5 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} f(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} k_u + k_0 & k_0 r_1 & 0 & 0 & 0 & 0 \\ k_0 r_1 & (k_0 + k_1) r_1^2 & -k_1 r_1 r_2 & 0 & 0 & 0 \\ 0 & -k_1 r_1 r_2 & (k_1 + k_2) r_2^2 & -k_2 r_2 r_3 & 0 & 0 \\ 0 & 0 & -k_2 r_2 r_3 & (k_2 + k_3) r_3^2 & -k_3 r_3 r_4 & 0 \\ 0 & 0 & 0 & -k_3 r_3 r_4 & (k_3 + k_4) r_4^2 & -k_4 r_4 r_5 \\ 0 & 0 & 0 & 0 & -k_4 r_4 r_5 & k_5 r_5^2 \end{bmatrix} .$$

We may use these equations for single and double bass-reflex loudspeaker systems as well, where $n = 1$ for single and $n = 2$ for double bass-reflex systems. Stiffness matrix K is tri-diagonal matrix so that it is not difficult to calculate characteristic frequencies.

3.3 Deriving Equation of Motion of Standard MCAP-CR (Multiple-Chamber Aligned in Parallel Cavity Resonator)

Standard MCAP-CR is typical application of multiple degree of freedom cavity resonators as well as MCAS-CR. Equations of motion under free vibration condition were already described in MCAP001E. Now we include mass of vibration membrane and discuss forced vibration equations.

Fig.8 shows structure of standard MCAP-CR. Number of sub-chambers n is theoretically unlimited, though we see only three in Fig.8.

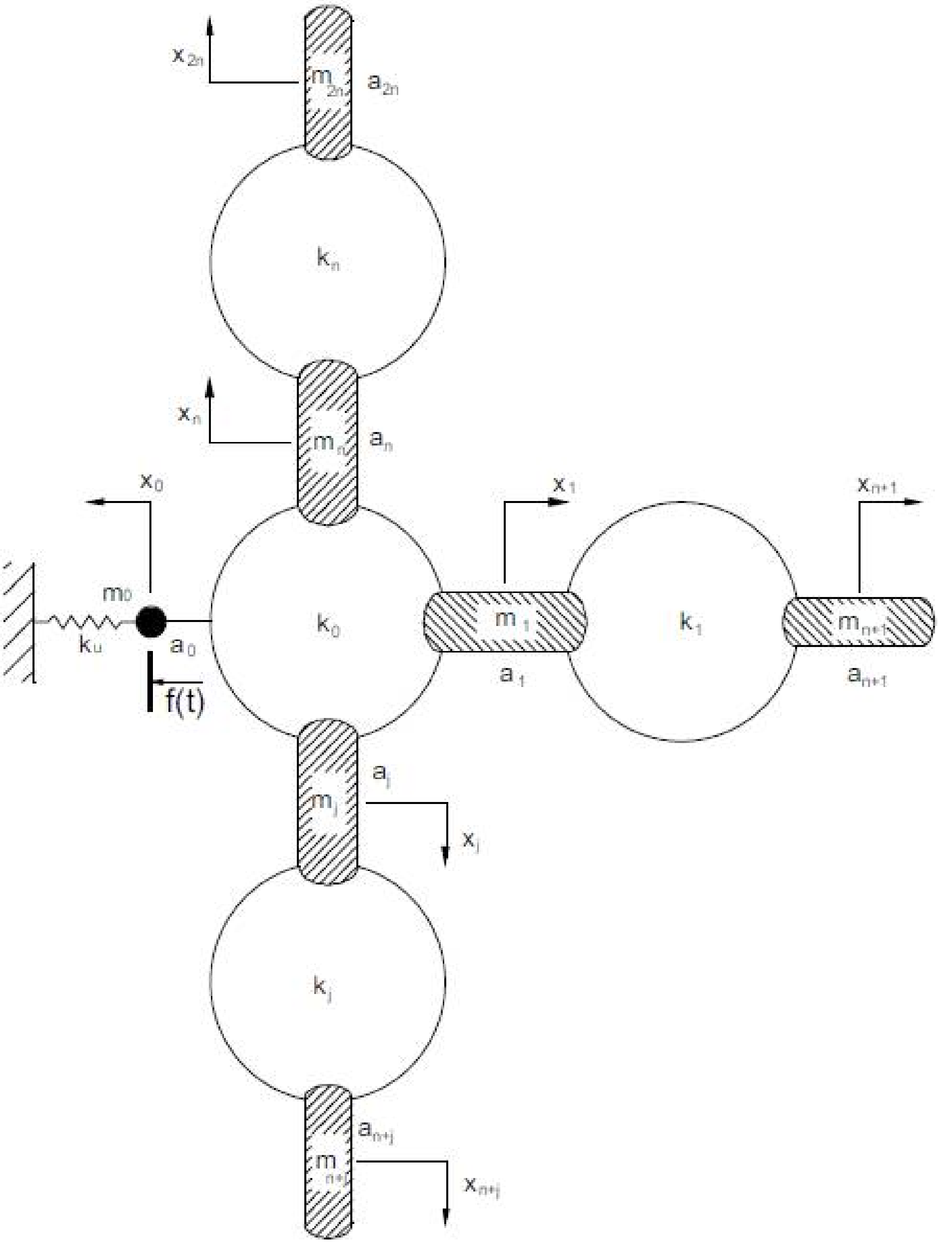

Fig.8 Structure of Standard MCAP-CR

Total restoring force to mass m_0 is derived adding restoring forces k_2 ,..., k_n of equation (8), so that we get equation of motion of mass m_0 as follows:

$$m_0 \frac{d^2 x_0}{dt^2} + (k_u + k_0) x_0 + k_0 r_1 x_1 + k_0 r_2 x_2 + \ldots + k_0 r_n x_n = f(t)$$
(12).

Equation of motion of mass m_j is derived in the same manner above.

$$m_j \frac{d^2 x_j}{dr^2} + k_0 r_0^2 \frac{a_j}{a_0} x_0 + k_0 r_1^2 \frac{a_j}{a_1} x_1 + \ldots + k_0 r_j^2 \frac{a_j}{a_j} x_j + \ldots + k_0 r_n^2 \frac{a_j}{a_n} x_n - k_j r_j^2 \frac{a_{n+j}}{a_j} x_j = 0$$

videlicet,

$$m_j \frac{d^2 x_j}{dt^2} + k_0 r_0 r_j x_0 + k_0 r_1 r_j + \ldots + (k_0 + k_j) r_j^2 x_j + \ldots + k_0 r_n r_j x_n - k_j r_k r_{n+j} x_{n+j} = 0$$
(13).

Equation if motion of mass m_{n+j} is expressed in equation (14), as same manner as MCAS-CR:

$$m_{n+j} \frac{d^2 x_{n+j}}{dt^2} - k_j r_j r_{n+j} + k_j r_{n+j}^2 x_{n+j} = 0 \tag{14}$$

Equation (12) applies to MCAP-CR same as MCAS-CR. For MCAP-CR, we get

$$\boldsymbol{M} = \begin{bmatrix} m_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_6 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} f(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where,

$$\boldsymbol{A} = \begin{bmatrix} k_u + k_0 & k_0 r_1 & k_0 r_2 & k_0 r_3 \\ k_0 r_1 & (k_0 + k_1) r_1^2 & k_0 r_1 r_2 & k_0 r_1 r_3 \\ k_0 r_2 & k_0 r_1 r_2 & (k_0 + k_2) r_2^2 & k_0 r_2 r_3 \\ k_0 r_3 & k_0 r_1 r_3 & k_0 r_2 r_3 & (k_0 + k_3) r_3^2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ -k_1 r_1 r_4 & 0 & 0 \\ 0 & -k_2 r_2 r_5 & 0 \\ 0 & 0 & -k_3 r_3 r_6 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & -k_1 r_1 r_4 & 0 & 0 \\ 0 & 0 & -k_2 r_2 r_5 & 0 \\ 0 & 0 & 0 & -k_3 r_3 r_6 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} k_1 r_4^2 & 0 & 0 \\ 0 & k_2 r_5^2 & 0 \\ 0 & 0 & k_3 r_6^2 \end{bmatrix}.$$

3.4 AICC-CR (Arbitrary Inter Chamber Connection Cavity Resonator) System

AICC-CR is the most complex system so that it is not easy to show typical application. We derive equations for Fig.5c of MCAP004E.

Fig.5c is similar to standard MCAP-CR with two sub-chambers where sub-chambers are connected each other. Therefore equation of motion for m_0 is same as equation (12) where $n=2$.

Equations of motion for m_1 and m_2 are as follows:

$$m_1\frac{d^2x_1}{dt^2}+k_0r_1^2x_1+k_0r_1r_2x_2+k_0r_1r_0x_0-k_1r_1r_3x_3-k_1r_1r_5x_5 \quad (15)$$

$$m_2\frac{d^2x_2}{dt^2}+k_0r_2^2x_2+k_0r_2r_1x_1+k_0r_2r_0x_0-k_2r_2r_4x_5-k_2r_2r_6x_6 \quad (16).$$

Equations of motion for m_3 and m_4 are as follows:

$$m_3\frac{d^2x_5}{dt^2}+k_1r_3^2x_1+k_1r_5r_3x_5-k_1r_1r_3x_1=0 \quad (17)$$

$$m_4\frac{d^2x_5}{dt^2}+k_2r_4^2x_4+k_2r_5r_4x_5-k_2r_2r_4x_4=0$$

(18).

Equation of motion for M_5 is as follows:

$$m_5\frac{d^2x_5}{dt^2}=\left(k_1+k_2\right)r_5^2+k_1r_5^2\left(\frac{a_3}{a_5}x_3-\frac{a_1}{a_5}x_1\right)+k_2r_5^2\left(\frac{a_2}{a_5}x_2-\frac{a_4}{a_5}x_4\right)=0 \quad .$$

Arranging above equation, we get

$$m_5\frac{d^2x_5}{dt^2}-k_1r_5r_1x_1+k_2r_5r_2x_2+k_1r_5r_3x_3-k_2r_5r_4x_4+\left(k_1+k_2\right)r_5^2x_5=0 \quad (18).$$

Deriving equations of motion for general AICC-CR application is not practical, since it is too complex. On the other hand, we can derive equations for specified AICC-CR application as seen above.

AICC-CR is, at this time , not well analyzed. We will later make some typical AICC-CR applications practical enough.

End of this report